A Note on He's Parameter-Expansion Method of Coupled Van der Pol–Duffing Oscillators

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Abstract

This paper presents the analytical and approximate solutions of the coupled chaotic Van der Pol-Duffing systems, by using the He's parameter-expansion method (PEM). One iteration is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain. From the obtained results, we can conclude that the suggest method, is of utter simplicity, and can be easily extended to all kinds of non-linear equations.

1. Introduction

In recent years, extensive investigations have been carried out to analyze chaotic synchronization dynamics of coupled non-linear oscillators Boccaletti et al. (2002), Bowong et al. (2006). The interest devoted to such topics is due to the potential applications of synchronization in

communication engineering (using chaos to mask the information signals Pecora and Caroll (1990), Perez and Cedeira (1995), electrical and automation engineering, biology, and chemistry (Kuramoto (1984), Winfree (1980).

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. There are many analytical methods and numerical schemes to handle non-linear problems, such as variational iteration method He (2006), Sweilam and Khader (2007), Sweilam et al. (2008), homotopy perturbation method He (2006) Turgut and Yildirim (2007), the exp-function method He and Wu (2006), standard Lindstedt-Poincaré He (2002) and parameter-expansion method (Darvishi et al. (2008), Xu (2007), Sweilam and Khader (2009). The parameter-expansion method (PEM) which proposed by He is proved to be a very effective and convenient way for handling the non-linear problems.

The organization of the paper is as follows. The next Section deals with the model and the problem statement. In Section 3, the application of the He's parameter-expansion method for solving the coupled chaotic Van der Pol-Duffing systems is presented. A conclusion is given in the last section.

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2.   Problem Statement

In the chaotic state, the main feature is the high-sensitivity to initial conditions. This is the result of the combined effects of the cubic and intrinsic nonlinearities, and of the extrinsic periodic drive. Consequently, a very small difference in the initial conditions will lead to different time histories or orbits. If two systems are launched with two initial conditions IC_1 and IC_2, they will circulate on different degenerated chaotic orbits. The goal of the synchronization in this case is to call one of the systems (slave) from its degenerated chaotic orbit to that of the other system (master). For this aim, the master system is described by the component x while the slave system has the corresponding component y. The enslavement is carried out by coupling the slave to the master through the following scheme Yamapi and Filatrella (2008):

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x + \alpha x^3 = E_0 \cos(\varepsilon t),$$
$$\ddot{y} - \mu(1 - y^2)\dot{y} + y + \alpha y^3 = E_0 \cos(\varepsilon t) - K(y - x)H(t - T_0), \tag{1}$$

(dots denote differentiation with respect to time). The quantities μ and α are two positive coefficients. E_0 and ε are respectively the amplitude and frequency of the external excitation. Where K is the feedback coupling coefficient, T_0 the onset time of the synchronization process and $H(z)$ is the Heaviside function defined as:

$$H(z) = \begin{cases} 0, & \text{for } z < 0; \\ 1, & \text{for } z \geq 0. \end{cases}$$

Thus, for $t < T_0$, the two oscillators are completely uncoupled; at $t = T_0$ the coupling turns on and stays on for all subsequent t. We let the system evolve unperturbed before the coupling is turned on to avoid a transient that could influence the beginning of the synchronization process. This will be relevant in the next session where we have estimated the synchronization time, not just the asymptotic properties of the system. Practically, this type of unidirectional coupling between the master system and the slave system can be done through a linear resistor and a buffer. The buffer acts a signal-driving element that isolates the master system variable from the slave system variable, thereby providing a one-way coupling. In the absence of the buffer the system represents two identical Van der Pol–Duffing oscillators coupled by a common resistor R_c, when both the master and slave systems will mutually affect each other. It is worth to note that the sign of K has an intuitive meaning: $K > 0$ corresponds to the attractive case between the master and the slave, while $K < 0$ corresponds to repulsion. A natural expectation would then be: for some value of positive K the system achieves synchronization, while for negative values of K one would not expect synchronization.

3.   Implementation of He's PEM

In this section, the basic idea of the parameter-expansion method will be illustrates by solving the coupled system of nonlinear ordinary differential equations (1). To implement PEM, we re-write the system (1) in the following form:

$$\ddot{x} + 1.x - \mu.(1 - x^2)\dot{x} + \alpha.x^3 = E_0.\cos(\varepsilon t),$$

$$\ddot{y} + (1 + K).y - \mu.(1 - y^2)\dot{y} + \alpha.y^3 - K.x = E_0.\cos(\varepsilon t). \tag{2}$$

According to PEM, the solutions and coefficients $1, (1 + K), \mu, \alpha, K$ and E_0 of (2) are expanded into a series of an artificial parameter, p respectively, in the forms:

$$x = x_0 + px_1 + p^2 x_2 + ...,$$
$$y = y_0 + py_1 + p^2 y_2 + ..., \tag{3}$$

$$1 = r^2 + p\,r_1 + p^2 r_2 + ...,$$
$$1 + K = s^2 + ps_1 + p^2 s_2 + ...,$$
$$\mu = pa_1 + p^2 a_2 + ...,$$
$$\alpha = pb_1 + p^2 b_2 + ..., \tag{4}$$
$$K = pc_1 + p^2 c_2 + ...,$$
$$E_0 = pd_1 + p^2 d_2 + ...$$

Now, by substituting from (3)-(4) into (2) respectively, and equating the terms with the identical powers of p, we can obtain a series of linear equations. These linear equations are easy to solving by using Mathematica software. Here we only write the first few linear equations:

$$\ddot{x}_0 + r^2 x_0 = 0, \qquad \dot{x}_0(0) = 0, \quad x_0(0) = 1,$$
$$\ddot{y}_0 + s^2 y_0 = 0, \qquad \dot{y}_0(0) = 0, \quad y_0(0) = 1,$$

(5)

$$\ddot{x}_1 + r^2 x_1 + r_1 x_0 - a_1(1 - x_0^2)\dot{x}_0 + b_1 x_0^3 - d_1 cos(\varepsilon t) = 0, \qquad \dot{x}_1(0) = x_1(0) = 0,$$
$$\ddot{y}_1 + s^2 y_1 + s_1 y_0 - a_1(1 - y_0^2)\dot{y}_0 + b_1 y_0^3 - c_1 x_0 - d_1 cos(\varepsilon t) = 0, \ \dot{y}_1(0) = y_1(0) = 0.$$

(6)

By considering the initial conditions, the solutions of (5) are:

$$x_0(t) = \cos(rt), \quad y_0(t) = \cos(st).$$

(7)

By substituting the result into the first equation of the system (6), we have:

$$\ddot{x}_1 + r^2 x_1 + r_1 \cos(rt) + a_1 r \sin^3(rt) + b_1 \cos^3(rt) - d_1 \cos(\varepsilon t) = 0,$$

or

$$\ddot{x}_1 + r^2 x_1 + (r_1 + \frac{3}{4} b_1)\cos(rt) + \frac{1}{4} b_1 \cos(3rt) + \frac{3}{4} a_1 r \sin(rt) - \frac{1}{4} a_1 r \sin(3rt) - d_1 \cos(\varepsilon t) = 0, \quad (8)$$

No secular term in x_1 requires that:

$$r_1 + \frac{3}{4} b_1 = 0.$$

(9)

If the first-order approximation is enough, then, setting p = 1 in Eqs. (3) - (4), we have:

$$x = x_0 + x_1, \quad 1 = r^2 + r_1, \quad \mu = a_1, \quad \alpha = b_1, \quad E_0 = d_1.$$

(10)

Solving the equations (9) – (10) simultaneously, we have:

$$r = \frac{1}{2}\sqrt{4 + 3\alpha}.$$

(11)

The approximate period can be expressed in the form:

$$T = \frac{2\pi}{r} = \frac{4\pi}{\sqrt{4 + 3\alpha}}.$$

(12)

To obtain $x_1(t)$, from the relations (8) - (11) we have:

$$\ddot{x}_1 + r^2 x_1 = -\frac{\alpha}{4}\cos(3rt) - \frac{3\mu r}{4}\sin(rt) + \frac{\mu r}{4}\sin(3rt) + E_0\cos(\varepsilon t), \tag{13}$$

Solving (13) yields:

$$\begin{aligned}
x_1(t) = &-(\frac{\alpha}{32r^2} + \frac{E_0}{r^2 - \varepsilon^2})\cos(rt) - \frac{9\mu}{32r}\sin(rt) + \frac{\alpha}{32r^2}\cos(3rt) + \frac{3\mu}{8}t\cos(rt) \\
&- \frac{\mu}{32r}\sin(3rt) + \frac{E_0}{r^2 - \varepsilon^2}\cos(\varepsilon t),
\end{aligned} \tag{14}$$

from the relations (7), (10) and (14), the first order solution for x is:

$$\begin{aligned}
x(t) = &\cos(rt) - (\frac{\alpha}{32r^2} + \frac{E_0}{r^2 - \varepsilon^2})\cos(rt) - \frac{9\mu}{32r}\sin(rt) + \frac{\alpha}{32r^2}\cos(3rt) \\
&+ \frac{3\mu}{8}t\cos(rt) - \frac{\mu}{32r}\sin(3rt) + \frac{E_0}{r^2 - \varepsilon^2}\cos(\varepsilon t),
\end{aligned} \tag{15}$$

where r is defined in (11).

Also, by substituting the result into the second equation of the system (6), we have:

$$\ddot{y}_1 + s^2 y_1 + s_1\cos(st) + sa_1\sin^3(st) + b_1\cos^3(st) - c_1\cos(st) - d_1\cos(\varepsilon t) = 0,$$

or

$$\ddot{y}_1 + s^2 y_1 + (s_1 + \frac{3}{4}b_1 - c_1)\cos(st) + \frac{3sa_1}{4}\sin(st) - \frac{sa_1}{4}\sin(3st) + \frac{b_1}{4}\cos(3st) - d_1\cos(\varepsilon t) = 0. \tag{16}$$

No secular term in y_1 requires that:

$$s_1 + \frac{3}{4}b_1 - c_1 = 0. \tag{17}$$

If the first-order approximation is enough, then, setting $p = 1$ in equations (3) - (4), we have:

$$y = y_0 + y_1, \quad 1 + K = s^2 + s_1, \quad \mu = a_1, \quad \alpha = b_1, \quad K = c_1, \quad E_0 = d_1. \tag{18}$$

By solving equations (17)–(18) simultaneously, we have:

$$s = \frac{1}{2}\sqrt{4+3\alpha}. \tag{19}$$

The approximate period can be expressed in the form (12).

To obtain $y_1(t)$, from relations (16)-(19) we have:

$$\ddot{y}_1 + s^2 y_1 = -\frac{\alpha}{4}\cos(3st) - \frac{3\mu s}{4}\sin(st) + \frac{\mu s}{4}\sin(3st) + E_0\cos(\varepsilon t), \tag{20}$$

solving (20) yields:

$$y_1(t) = -(\frac{\alpha}{32s^2} + \frac{E_0}{s^2 - \varepsilon^2})\cos(st) - \frac{9\mu}{32s}\sin(st) + \frac{\alpha}{32s^2}\cos(3st) + \frac{3\mu}{8}t\cos(st)$$
$$- \frac{\mu}{32s}\sin(3st) + \frac{E_0}{s^2 - \varepsilon^2}\cos(\varepsilon t), \tag{21}$$

from the relations (7), (18) and (21), the first order solution for y is:

$$y(t) = \cos(st) - (\frac{\alpha}{32s^2} + \frac{E_0}{s^2 - \varepsilon^2})\cos(st) - \frac{9\mu}{32s}\sin(st) + \frac{\alpha}{32s^2}\cos(3st)$$
$$+ \frac{3\mu}{8}t\cos(st) - \frac{\mu}{32s}\sin(3st) + \frac{E_0}{s^2 - \varepsilon^2}\cos(\varepsilon t). \tag{22}$$

where *s* is defined in (19).

Figs. 1-2 Illustrate the behavior of the solution (x(t), y(t)) with different values of the coefficients μ, α, ε, and E_0 in the system (1).
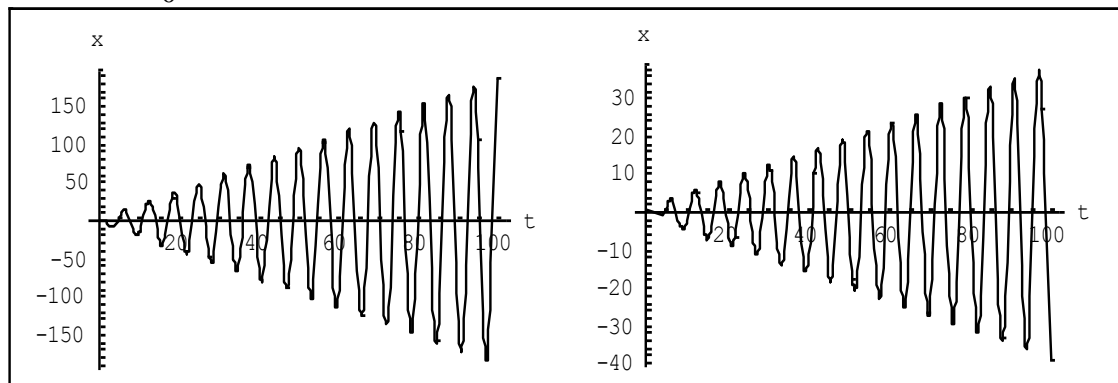

Figure 1:   Left: μ=5;   α=0.01;   ε=2.463; E_0=5;

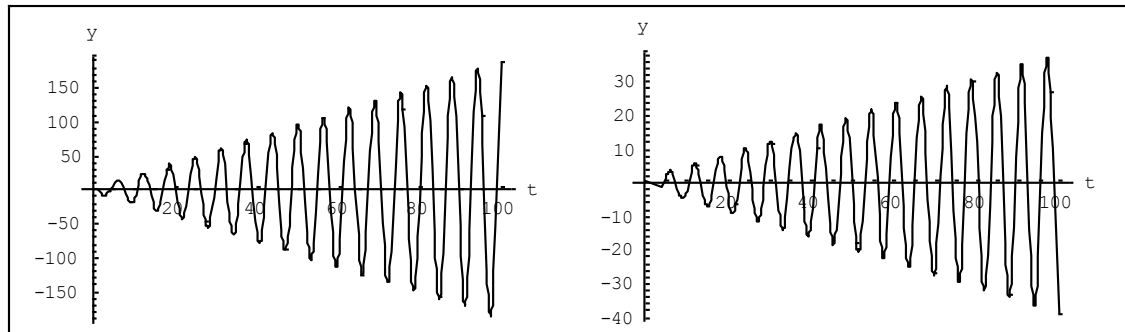**Right: μ=1;   α=0.1;   ε=2.463; E_0=1;**

Figure 2:   Left: $\mu=5$; $\alpha=0.01$; $\varepsilon=2.463$; $E_0=5$;

Right: $\mu=1$; $\alpha=0.1$; $\varepsilon=2.463$; $E_0=1$;

4.   Conclusion

From the presented results of the example in this paper, we conclude that He's parameter-expansion method is an extremely simple and a powerful tool to solve this particular problem of mutually coupled biological systems described by coupled chaotic Van der Pol-Duffing systems. The method might become a promising and powerful new method for many other applications in searching for periodic solutions of non-linear oscillations without any difficulty. The suggest method by help of Mathematica, is of utter simplicity, and can be easily extended to all kinds of non-linear equations. The PEM can be easily comprehended with only a basic knowledge of advanced calculus.

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